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Mastery of Mathematical Induction among Junior College Students[1]

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Abstract: *Junior college students often experience difficulty with mathematical induction. In this study, the data collection involved the use of proof-writing tasks in questionnaires followed by interviews of selected students. With respect to students' performance, they had difficulty with questions that involved formulating a conjecture and then proving it using mathematical induction. The junior college students had significant difficulties with the conceptual, procedural and technical aspects of the proof technique. The inability to complete a proof was also attributed to a lack of specific mathematical content knowledge. From the study, some good mathematical skills exhibited by good students were observed.*

Introduction

Mathematical induction is part of the mathematics syllabus required for the Singapore-Cambridge General Certificate in Education Advanced Level Examinations. Even after the content reduction by the Ministry of Education, Singapore, mathematical induction is still relevant in the revised syllabus for year 2000 and beyond (University of Cambridge Local Examinations Syndicate, 2000). For 16 to 17 year olds in the junior colleges, this topic is usually covered in their first year. Mathematical induction has always been a topic these students find difficult to understand. Teachers also experience difficulty explaining the concept of mathematical induction to students. Teachers have observed from tutorials, tests and examinations that students' ability to do problems in mathematical induction depends on the types of identities that are to be proved (Baker, 1996; Dubinsky, 1989). It is the intention of this study to focus on such difficulties.

The following quote highlights the importance of mathematical induction in the school curriculum: "A third goal is to increase attention to proof by mathematical induction, the most prominent proof technique in discrete mathematics" (National Council of Teachers of Mathematics (NCTM), 1989, p. 143). When taught well, mathematical induction can improve students' understanding of these methods. In *Principles and Standards for School Mathematics*, NCTM further recommended that "since iteration and recursive methods are increasingly common, students

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should learn that certain types of results are proved using the technique of mathematical induction" (NCTM, 2000, p. 345).

Previous researchers (Avital & Libeskind, 1978; Dubinsky, 1986, 1989; Ernest, 1984; Fischbein & Engel, 1989; Foret, 1998) agreed that mathematical induction is a very difficult concept to master. Avital and Libeskind (1978) described in detail the pedagogical problems and misconceptions which occurred while students learned mathematical induction. The kinds of difficulties they highlighted included (a) conceptual, (b) mathematical, and (c) technical where conceptual difficulties meant difficulties related to the implication from P_k to P_{k+1} and difficulty getting used to the transition from k to $k+1$, mathematical difficulties generally meant difficulties related to basis case not equal to one, and technical difficulties to difficulties related to the interpretation of the induction step applied in a particular problem and difficulties in the algebraic manipulations involved in proving the induction step.

Ernest (1984) also observed that many students had difficulties in producing correct proofs by the method of mathematical induction. His analysis of the topic of mathematical induction was in terms of behavioural skills and conceptual understanding. The three behavioural skills analysed were (a) the ability to prove the basis of the induction, (b) the ability to prove the induction step, and (c) the ability to present a proof by mathematical induction in the correct form. The two major conceptual clusters revealed were (a) defined properties of natural numbers and functions and (b) recurrence and the ordering of the natural numbers. Dubinsky (1989) taught mathematical induction using reflective abstraction to 24 students from the University of California at Berkeley and 16 students from Clarkson University. The results showed that the students performed very well for questions involving summation and algebraic identity. However, the question that involved formulating a conjecture and then proving it was one of the most difficult for the students.

The teaching of mathematical induction and students' learning difficulties in this topic are still attracting attention from researchers (Allen, 2001; Baker, 1996; Movshovitz-Hadar, 1993; Murakami, 2000). Baker (1996) investigated 40 high school and 13 college students as they began to learn mathematical induction. Students provided data in the form of proof-writing and proof-analysis tasks followed by interviews to clarify their responses. Baker concluded that many students focused on the procedural aspects of mathematical induction far more often than on conceptual aspects. From that study, Baker suggested that specific mathematical content knowledge like summation symbol, factorial symbol, definition of variable and algebra played a significant role in difficulties.

Therefore, based on this background (and as no Singapore-based local research has been done based on the syllabus required for the Singapore-Cambridge General Certificate in Education Advanced Level Examinations to identify the types of questions on mathematical induction that Singapore students can do, the kinds of questions where they tend to falter, and the reasons behind their weaknesses), it was the purpose of this study to investigate the types of difficulties Singapore students face in the learning of mathematical induction.

Given the wide range of usage of terms like conceptual, mathematical and technical difficulties by different authors, there is considerable variation in the meaning of these terms. For this study, the following framework of understanding of terms was adopted. Difficulties were classified into conceptual, procedural and technical. Conceptual difficulties refer to the understanding of the method of proof by mathematical induction, the relationship between inductive hypothesis and inductive step and the need for the basis case. Procedural difficulties refer to difficulties encountered when basis case does not start at $n=1$ and giving the wrong proposition statement. Technical difficulties refer to difficulties in simplifying algebraic fractions and incorrect substitution of variables. As well, very often, the inability to complete a proof by mathematical induction could be due to a lack of specific mathematical content knowledge by the students. A separate category on lack of mathematical content knowledge like indices, sigma notation sequence and factorial was created.

The research questions were:

1. What is the relationship between students' performance in mathematical induction and the types of identities to be proved?
2. What are the kinds of conceptual, procedural and technical difficulties that students encounter when they learn mathematical induction?

Research Methodology

Subjects

This study was conducted in the second half of 2000 in a junior college located in the eastern part of Singapore with a total student population of around 1600. Eighty-six year one students (17 years of age) answered this questionnaire. These students were picked at random from students taking double mathematics (both Further Mathematics and Mathematics 'C'), Science students taking Mathematics 'C' only, and Arts students taking Mathematics 'C' only. The mathematics backgrounds of these students thus differed widely. From the data collected, not all of the 86 students attempted all the questions in the questionnaire. Those questionnaires with certain questions not attempted or with very short and mostly incomplete workings were discarded. As a result, only 30 questionnaires were selected for data analysis.

Thus the sample consists of the best and most conscientious students of the cohort. These selected 30 students had a mean L1R5 of 12.93 with a range of 8 to 16. Their average grades for Elementary Mathematics and Additional Mathematics at General Certificate in Education Ordinary Level were 1.47 and 2.50 respectively.

Instrumentation

The proof-writing questionnaire was developed by examining various sources:

(a) Test items from past research articles,
(b) Past-year questions from General Certificate in Education Advanced Level Examinations, and
(c) Past-year Preliminary Examination questions from various junior colleges.

Based on the various types of identities tested in past General Certificate in Education Advanced Level Examinations, the questionnaire comprised five proof-writing questions on the identities involving (a) indices, (b) the summation symbol, (c) sequences, (d) differentiation of one variable and factorial symbol, and (e) formulation and subsequent proof of a conjecture. The questions in the questionnaire are reproduced in Table 1. The summation sign appeared in both questions 1 and 2 and sequence was used in both questions 3 and 5. However, labeling of the five questions related to the anticipated difficulties encountered by students.

Data collection

Data collection involved two aspects: a questionnaire and interviews. The questionnaire consisting of five proof-writing questions was administered to the subjects under examination conditions. They were not told in advance about the questionnaire. The subjects did the questionnaire during one of their tutorial sessions in a classroom and were given 45 minutes to complete it. The duration of the test was within one tutorial session of 45 minutes to parallel a typical tutorial session activity.

Based on the questionnaires, seven students were selected for interviews to clarify and elaborate on their written answers. Their selection was determined as follows: (a) lowest mark in any one question in proof-writing questionnaire, (b) highest mark in any one question in proof-writing questionnaire, and (c) answers that were ambiguous or intriguing. All the interviews were conducted in an empty classroom. Individual interviews were audio taped for subsequent analysis. Before the commencement of the interview, subjects were given time to look through their earlier written responses. The subjects had no idea whether their written proofs were correct or not as only the codes were given. During the interviews, the subjects were not informed if their replies were correct.

Table 1
Questionnaire on mathematical induction

1. Show by induction that, for every positive integer n, $$\sum_{r=1}^{n} \frac{r+1}{2^r} = 3 - \left(\frac{1}{2}\right)^n (n+3).$$
2. Prove by induction that $$\sum_{r=0}^{n} \frac{1}{(r+1)(r+3)(r+5)} = \frac{11}{96} - \frac{1}{4(n+2)(n+4)} - \frac{1}{4(n+3)(n+5)}$$
3. Given that $a_1 = 1$ and $a_{n+1} = 3a_n + 2n - 2 \quad (n \geq 1, n \in Z^+)$. Prove, by induction that $a_n = \frac{1}{2}\left(3^n + 1\right) - n$ for all positive integers n.
4. Given that $y = \frac{1}{1+x}$, prove by induction that $\frac{d^n y}{dx^n} = \frac{(-1)^n (n!)}{(1+x)^{n+1}}$ for every positive integer n.
5. Let (x_n) be a sequence of numbers defined by: $x_1 = 1$, $x_{n+1} = 2^n x_n$ for $n = 1,2,...$ Find a formula for x_n and prove it by mathematical induction.

Scoring and data analysis
Written answers were analyzed in two categories:
(a) Students' performance in different proof-writing tasks, and
(b) Students' difficulties with this proof technique, namely: conceptual, procedural and technical difficulties.

Student performance for the proof-writing tasks was evaluated with a marking scheme given in Table 2. Each of the five questions was scored from 0 through 10.

The average scores of the 30 students for each of the five questions was tabulated. A lower score for a question represented a higher level of difficulty for that type of question. The five scores were ranked in descending order to identify the students' performance in different proof-writing tasks.

The questionnaires were then analyzed for student errors and difficulties using the proposed framework in terms of conceptual, procedural and technical difficulties. Number codes were given to different learning difficulties. The frequency of occurrence and percentages for each category of difficulties were calculated.

Results and Related Discussion

Students' performance

The first research question which focused on the relationship between students' performance in mathematical induction and the types of identities to be proved was investigated. This relationship is illustrated in Table 3. Students in this study did well with proof involving indices and summation. From Table 3, the average score of 30 students in proof involving indices (Question 1) was 5.7 out of possible 10.

Table 2
Marking scheme for proof-writing tasks

Descriptions	Mark
• The statement to be proved to be represented by some notation as P_n.	1
• Clear verification that the result is true for the appropriate initial value.	1
• Clear statement of the general case that is being assumed (inductive hypothesis).	1
• A statement of the result (P_{k+1}) that is going to be deduced before the main algebraic part of the proof in the induction step begins.	1
• Algebra in inductive step must include enough working to show clearly that the expected result has in fact been properly derived.	5
• Conclusion to round off the proof.	1

Table 3
Types of questions and students' score. (Maximum score = 10. Sample size = 30)

Question	Label	Mean Score	Standard Deviation
1	Indices	5.70	2.49
2	Summation	5.67	3.69
3	Sequences	5.50	2.69
4	Differentiation and factorial	5.30	3.20
5	Formulate a conjecture	0.80	2.22

Proofs involving sequences, differentiation and factorial symbol were not as easy for the students. The most difficult question involved formulating a conjecture and then proving it using mathematical induction. From Table 3, only an average score of 0.8 was recorded for question 5.

From the proof-writing tasks, it can be observed that students' success in the induction step depended on their ability to write down the key equation that linked the propositions P_k to P_{k+1}. The key equation in each of the five questions is provided in Table 4. From Table 4, questions 3 and 4 were difficult because the way to obtain the linking equations to prove the induction steps was different from that in questions 1 and 2. Questions 1 and 2 involve summation signs. To link P_k to $P_{k+1,}$ students must obtain the (k+1)th term. They were trained to substitute the variable r in the general term in the sigma notation by k+1. The induction step for questions 1 and 2 involved adding expression in P_k to (k+1)th term. The students also presupposed the same method in question 3 and question 4 which involved sequences and differentiation. In question 3, three students wrote down the equation that linked P_k to P_{k+1} as $a_{k+1} = a_k + a_{k+1}$. The (k+1)th term was a_{k+1}. They substituted a_k as $a_k = \frac{1}{2}\left(3^k + 1\right) - k$ using the inductive hypothesis. The term a_{k+1} was replaced by the iteration formula: $a_{k+1} = 3a_k + 2k - 2$. Part of a student's working for question 3 is reproduced in Written Sample 1.

<u>Written Sample 1: TYY</u>

Assume that P_k is true for some integers $k \ge 1$

i.e. $a_k = \frac{1}{2}\left(3^k + 1\right) - k$

Table 4

Table showing the key equation that links P_k to P_{k+1}

Question	Key equation that links P_k to P_{k+1}
1	$\sum_{r=1}^{k+1}\frac{r+1}{2^r} = \sum_{r=1}^{k}\frac{r+1}{2^r} + \frac{(k+1)+1}{2^{k+1}}$
2	$\sum_{r=0}^{k+1}\frac{1}{(r+1)(r+3)(r+5)} =$ $\sum_{r=0}^{k}\frac{1}{(r+1)(r+3)(r+5)} + \frac{1}{(k+2)(k+4)(k+6)}$
3	$3a_k + 2k - 2 \quad = 3\left[\frac{1}{2}(3^k + 1) - k\right] + 2k - 2$
4	$\frac{d^{k+1}y}{dx^{k+1}} = \frac{d}{dx}\left(\frac{d^k y}{dx^k}\right)$
5	$x_{k+1} = 2^k x_k$

<u>Written Sample 1: TYY (con'd)</u>

Want to prove that P_{k+1} is true

i.e. $a_{k+1} = \frac{1}{2}(3^{k+1} + 1) - (k+1)$

Given that $a_{k+1} = 3a_k + 2k - 2$

Then, $a_k + a_{k+1} = \frac{1}{2}(3^k + 1) - k + 3a_k + 2k - 2$

The same difficulty of presupposing that $a_{k+1} = a_k + a_{k+1}$ was also revealed during an interview with a student (QPB) on question 3, reproduced in Vignette 1.

Vignette 1: QPB

I: How could you show that P_k is true implies P_{k+1} is true?
QPB: I used the equation $a_{k+1} = a_k + a_{k+1}$.
I: Why did you use this equation $a_{k+1} = a_k + a_{k+1}$?
QPB: a_{k+1} is a_k plus the (k+1)th term.
I: Do you mean that your (k+1)th term is a_{k+1}?
QPB: Yes.
I: Did you realize that a_{k+1} appeared on both sides of your equation?
QPB: ...

Table 5 records the number of students, by question, who could not write down the key equation. The table also shows the number of students who were able to write down the equation that provided the link but were unable to complete the induction step. From Table 5, in question 4, 15 out of 30 students had difficulty writing down the key equation and 7 students could link but could not complete the induction step. Questions 4 and 5 seemed to be the most difficult questions. As for question 4, 6.7%, or 2 out of 30, of the students had the misconception that $\frac{d^{k+1}y}{dx^{k+1}} = \frac{d^k y}{dx^k} + \frac{dy}{dx}$.

Table 5
Student's performance based on ability to write down the key equation that links P_k to P_{k+1}

Question	1	2	3	4	5
Cannot link P_k to P_{k+1}	3	7	6	15	27
Can link but cannot complete the induction step	15	12	14	7	1

The operation is not addition but a process of differentiation. Part of a student's working for question 4 is reproduced in Written Sample 2.

Written Sample 2: OHS

$$\text{LHS} = \frac{dy}{dx} = \frac{-1}{(1+x)^2}$$

$$\text{RHS} = \frac{(-1)^1(1!)}{(1+x)^{1+1}}$$

$\therefore P_1$ is true.

Assume that P_k is true for some $k \in Z^+$

i.e. $\frac{d^k y}{dx^k} = \frac{(-1)^k (k!)}{(1+x)^{k+1}}$

Want to prove that $\frac{d^{k+1} y}{dx^{k+1}} = \frac{(-1)^{k+1} (k+1)!}{(1+x)^{k+2}}$

LHS = $\frac{(-1)^k (k!)}{(1+x)^{k+1}} + \frac{-1}{(1+x)^2}$

This misconception, that $\frac{d^{k+1} y}{dx^{k+1}} = \frac{d^k y}{dx^k} + \frac{dy}{dx}$, also surfaced during an interview with a student (HSY) on question 4, reproduced in Vignette 2.

Vignette 2: HSY

I: How did you show that P_k is true implies P_{k+1} is true?

HSY: I used the equation $\frac{d^{k+1} y}{dx^{k+1}} = \frac{d^k y}{dx^k} + \frac{dy}{dx}$.

I: Why did you use this equation?

HSY: ... P_{k+1} is P_k plus (k+1)th term.

I: Do you mean that your (k+1)th term is $\frac{dy}{dx}$?

HSY: ... I think so.

Types of difficulties

The second research question investigated the kinds of learning difficulties that students encounter when they learn mathematical induction. Critically, the junior college students had significant difficulties with the proof technique, either conceptually, procedurally or technically. A primary source of difficulty was attributed to a lack of mathematical content knowledge. Difficulties classified under conceptual and procedural are related specifically to mathematical induction. Technical difficulties and lack of content knowledge are related to prerequisite skills. A list of difficulties experienced by students is shown in Table 6. For example, under conceptual difficulties, five out of 30 students in the study assumed that inductive hypothesis is true for all *k*. In Written Sample 3, part of a student's working is shown.

Written Sample 3: PKK

Assume that P_k is true for all $k \geq 1$.

The correct form should be to assume that P_k is true for some $k \geq 1$. If the students assumed that the proposition was true for all integer values of *k*, then there was no need to prove the induction step. Before proving, the proposition was already assumed true. Students were inclined to consider the absolute truth value of the inductive hypothesis. Fischbein and Engel (1989) pointed out that this difficulty lies in the students having to build on the entire segment of the induction step based on an inductive hypothesis which, itself, has not been proven.

Table 6
Significant types of students' learning difficulties

Difficulties	Descriptions	N
Conceptual difficulties	• Inductive hypothesis is true for all *k*	5
	• Unable to prove induction step $P_k \Rightarrow P_{k+1}$	8
	• Insufficient working in proving the basis case	2
Procedural difficulties	• Basis case is not *n*=1	10
	• Proposition P_n is false	1
Technical difficulties	• Negative sign in front of algebraic fraction	8
	• Sigma notation	5
	• Wrong substitution	1
Lack of mathematical content knowledge	• Indices	4
	• Sigma notation	3
	• Sequences notation	5
	• Differentiate a factorial which is a constant	5
	• Factorial symbol	2

N = number of students out of 30.

Procedurally, ten of the students in the study encountered difficulty when the basis case was not *n*=1. Question 2 was purposely selected to test this when the basis case was *n*=0. One third of the students still began the proof with the basis case of *n*=1. They were thrown into a "state of disequilibrium" when the value evaluated at *n*=1 for left-hand side and right-hand side were not equal. An interview with a student (YAL) shown in Vignette 3 also revealed this uncertainty.

<u>Vignette 3: YAL</u>

I: You proved the basis case using *n*=1 and your LHS=RHS. Your LHS is $\frac{1}{48}$. Please show me that your RHS is $\frac{1}{48}$.

YAL: Actually, my RHS= $\frac{7}{80}$.
I: That means your proof of LHS is not equal to RHS.
YAL: Yes. But I know I am supposed to show that LHS=RHS.
I: Then what went wrong?
YAL: ...

For technical difficulty, eight out of 30 students, or 26.7%, made mistakes simplifying an algebraic fraction with a negative sign in front. This is evident in the induction step for question 1. Part of a student's working is reproduced in Written Sample 4.

Written Sample 4: TKY

Assume that P_k is true for some integers $k \geq 1$.

$$\sum_{r=1}^{k} \frac{r+1}{2^r} = 3 - \left(\frac{1}{2}\right)^k (k+3)$$

Want to prove that P_{k+1} is true

i.e. $\sum_{r=1}^{k+1} \frac{r+1}{2^r} = 3 - \left(\frac{1}{2}\right)^{k+1} (k+4)$

Then, LHS $= \sum_{r=1}^{k} \frac{r+1}{2^r} + \frac{(k+1)+1}{2^{k+1}}$

$$= 3 - \left(\frac{1}{2}\right)^k (k+3) + \frac{1}{2}\left(\frac{1}{2}\right)^k (k+2)$$

$$= 3 - \left(\frac{1}{2}\right)^k \left[(k+3) + \frac{1}{2}(k+2)\right]$$

The sign in the bracket should be negative. The students knew that (-)(-) gives a (+) sign. They also knew that something was wrong when the simplification did not reach the target expression in the proposition P_{k+1}. However, the high occurrence of such a mistake may signal the need for other techniques to simplify negative algebraic fractions.

The inability to complete a proof was also due to a lack of specific mathematical content knowledge. In many cases, students were unable to operate with symbols and to use algebraic procedures. The ones that appeared in the proof-writing test were a subset of what the students might have mastered in secondary schools and in junior college. Five students out of 30 were unable to differentiate $\frac{(-1)^k (k!)}{(1+x)^{k+1}}$ with respect to *x*. The problem lies in the variable that they need to differentiate. From an interview with a student (LJX) on question 4, the student thought that k was a variable and thus differentiated with respect to k. A portion of the interview from LJX is reproduced in Vignette 4. The student could find the derivative of $(-1)^k$ with respect to *k*. However, *k* is a constant.

<u>Vignette 4: LJX</u>

I: Please explain to me how you perform $\frac{d}{dx}\frac{(-1)^k\, k\,!}{(1+x)^{k+1}}$.

LJX: I differentiate $(-1)^k$ to get $k\,(-1)^{k-1}$. Then I have to differentiate *k*!. But I don't know how to differentiate *k*!.

I: Do you differentiate with respect to *x* or *k*?

LJX: I see *k* in $(-1)^k$, so I differentiate the *k*.

From this study, good students were observed to have acquired the following good mathematical skills:

- Identify common factors
- Use commutative law to rearrange algebraic fractions
- Simplify target expression in P_{k+1}
- Isolate the constants
- Observe for pattern

In the induction step for question 2, the good students realised that $-\frac{1}{4(k+3)(k+5)}$ appeared in the target expression in P_{k+1}. This term was left untouched. Instead of using brackets to simplify $-\frac{1}{4(k+2)(k+4)}+\frac{1}{(k+2)(k+4)(k+6)}$ to get $-\frac{1}{4(k+2)(k+4)(k+6)}[(k+6)-4]$, students interchanged positions of the two terms using laws of commutativity to put the negative algebraic fraction behind. Part of a student's working for question 2 is reproduced in Written Sample 5.

Written Sample 5: WYJ

$$\text{LHS} = \sum_{r=0}^{k+1} \frac{1}{(r+1)(r+3)(r+5)}$$

$$= \sum_{r=0}^{k} \frac{1}{(r+1)(r+3)(r+5)} + \frac{1}{(k+1+1)(k+1+3)(k+1+5)}$$

$$= \frac{11}{96} - \frac{1}{4(k+2)(k+4)} - \frac{1}{4(k+3)(k+5)} + \frac{1}{(k+2)(k+4)(k+6)}$$

$$= \frac{11}{96} - \frac{1}{4(k+3)(k+5)} + \frac{1}{(k+2)(k+4)(k+6)} - \frac{1}{4(k+2)(k+4)}$$

$$= \frac{11}{96} - \frac{1}{4(k+3)(k+5)} + \frac{4-(k+6)}{4(k+2)(k+4)(k+6)}$$

Instead of just writing down the target expression, good students also simplified the expression. Once the simplified expression appeared in the proof, the students could in a way trace backwards to the target expression in $P_{k+1.}$ A sample of a student's working for question 3 is reproduced in Written Sample 6. The target expression $a_{k+1} = \frac{1}{2}\left(3^{k+1}+1\right) - (k+1)$ is simplified to $\frac{1}{2}\left(3^{k+1}\right) - \frac{1}{2} - k$.

Written Sample 6: LMP

Want to prove that $a_{k+1} = \frac{1}{2}\left(3^{k+1}+1\right) - (k+1) = \frac{1}{2}\left(3^{k+1}\right) - \frac{1}{2} - k$

then,

$$\begin{aligned}
\text{LHS} &= a_{k+1} \\
&= 3a_k + 2k - 2 \\
&= 3\left[\frac{1}{2}\left(3^k+1\right) - k\right] + 2k - 2 \\
&= \frac{3}{2}\left(3^k+1\right) - 3k + 2k - 2 \\
&= \frac{1}{2}\left(3^{k+1}\right) + \frac{3}{2} - k - 2 \\
&= \frac{1}{2}\left(3^{k+1}\right) - \frac{1}{2} - k
\end{aligned}$$

Implications for Teaching

The results of this study suggest some implications for teaching. Most students had difficulties with proofs involving differentiation and sequences. For differentiation, students need to acquire the concept of higher derivative in the general sense. They need to know that the (k+1)th derivative of a function is obtained by differentiating the kth derivative of the function one more time. For sequences, students need to understand the purpose of the iteration formula that generates the next consecutive term.

The most difficult question involved formulating a conjecture and then proving it by mathematical induction. Observing a pattern is important in formulating the conjecture. Wiscamb (1970) suggested that students should discover the identities for themselves before proving them by mathematical induction. This proposal was supported by Avital and Hansen (1976) who proposed the teaching of mathematical induction through inductive investigation and student involvement. Van Schalkwijk, Bergen and Van Rooij (2001) also advocated investigations as learning environments for proving.

Another problem that frequently arose during the inductive step of the proof was that students thought that they were assuming what they were actually trying to prove. The instructional treatment could probably be improved by developing more effective techniques for inducing the reflective abstractions described by Dubinsky and Lewin (1986). The three schemas that are assumed to be present when students begin to study induction are method of proof, function and logical necessity. For example, suppose P_1 is true. An evaluation of implication-valued function $P_n \Rightarrow P_{n+1}$ at n=1 obtains $P_1 \Rightarrow P_2$. Applying *modus ponens* and the fact that P_1 is true yields P_2 is true. The evaluation process with n=2 yields $P_2 \Rightarrow P_3$. This is repeated *ad infinitum*.

Students need to understand the relationship between the basis case, inductive hypothesis and induction step of mathematical induction. They need to realize why the proof is weak by considering only the inductive step ($P_k \Rightarrow P_{k+1}$) or the basis case alone. Concept images of mathematical induction have been presented by numerous authors to illustrate the relationship between the method of induction and the well ordering of the natural numbers (Ernest, 1984; Lowenthal & Eisenberg, 1992; Scott, 2000). These analogies included:

- the knocking down of a long line of dominoes,
- the ascent of a ladder, step by step, and
- the entry of a princess into all the locked rooms of a palace, given that she has the key to the first room and that each room contains the key to the next room.

Take, for example, in the knocking down of dominoes, a domino will fall if the preceding domino falls ($P_k \Rightarrow P_{k+1}$). However, all the dominoes will not fall (P_n is not true yet) unless the first domino falls (P_1 is true).

It is a common technique used in lectures that only correct solutions are given. Proof-analysis tasks may be incorporated in lectures and even in tests and examinations to test students' understanding. An explanation of errors or showing that such an approach would lead to errors may help students to understand the topic better. It may also alert the students to misconceptions, thus avoiding them. Further, both proof analysis and teacher demonstrated examples can help make up for a lack of student generated examples.

Since proof by mathematical induction requires sufficient mathematical content knowledge to be successful, teachers should not assume mastery of content knowledge when demonstrating examples or constructing examinations. Szombathelyi and Szarvas (1998) agreed that students must learn how to use symbols and how to express themselves in the language of mathematics when proving a statement by induction. Notations like summation, factorial symbols and sequences must be formally introduced and well taught before the introduction of mathematical induction. Initially, students need to see that $\sum_{r=1}^{k+1} r^3 = \underbrace{1^3 + 2^3 + 3^3 + ... + k^3}_{\sum_{r=1}^{k} r^3} + (k+1)^3$. Students also need to know which is constant and which is a variable: *k* or *n* or *r*? For questions involving differentiation, students need to know that $\frac{d}{dx}$ means 'differentiate with respect to *x*'. To improve on the lack of content knowledge, in spite of the additional time required, remedial worksheets to supplement textbook exercises may be designed to improve on manipulation of sigma notation, factorial symbol and algebraic identities. The good mathematical skills exhibited by good students should also be taught to other students so as to minimize errors in mathematical induction.

The ultimate goal is that proof by mathematical induction becomes an integrated part of a students' repertoire and that students are more or less able to decide when it might be useful to apply it in a given situation without having been instructed beforehand to "prove by mathematical induction".

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